Monte Carlo for Hypothesis Testing

Module DES130: Computational Statistics

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Hypothesis testing

Example: battery lifetime

A company that produces watch batteries need to ensure that the lifetime of their batteries has a population mean equals to 72 hours. The lifetime of 10 batteries were obtained through an experiment observing the following values:

31.94, 140.44, 39.12, 82.47, 20.64, 37.89, 1.14, 15.4, 121.91, 15.59.

Take a decision assuming that the observations come from a normal distribution with known standard deviation σ = 47.8.

Example: battery lifetime

Hypothesis:

- (H_0): The population mean is equal to 72 ($\mu = 72$).
- (H_1): The population mean is not equal to 72 ($\mu \neq 72$).

Test statistic:

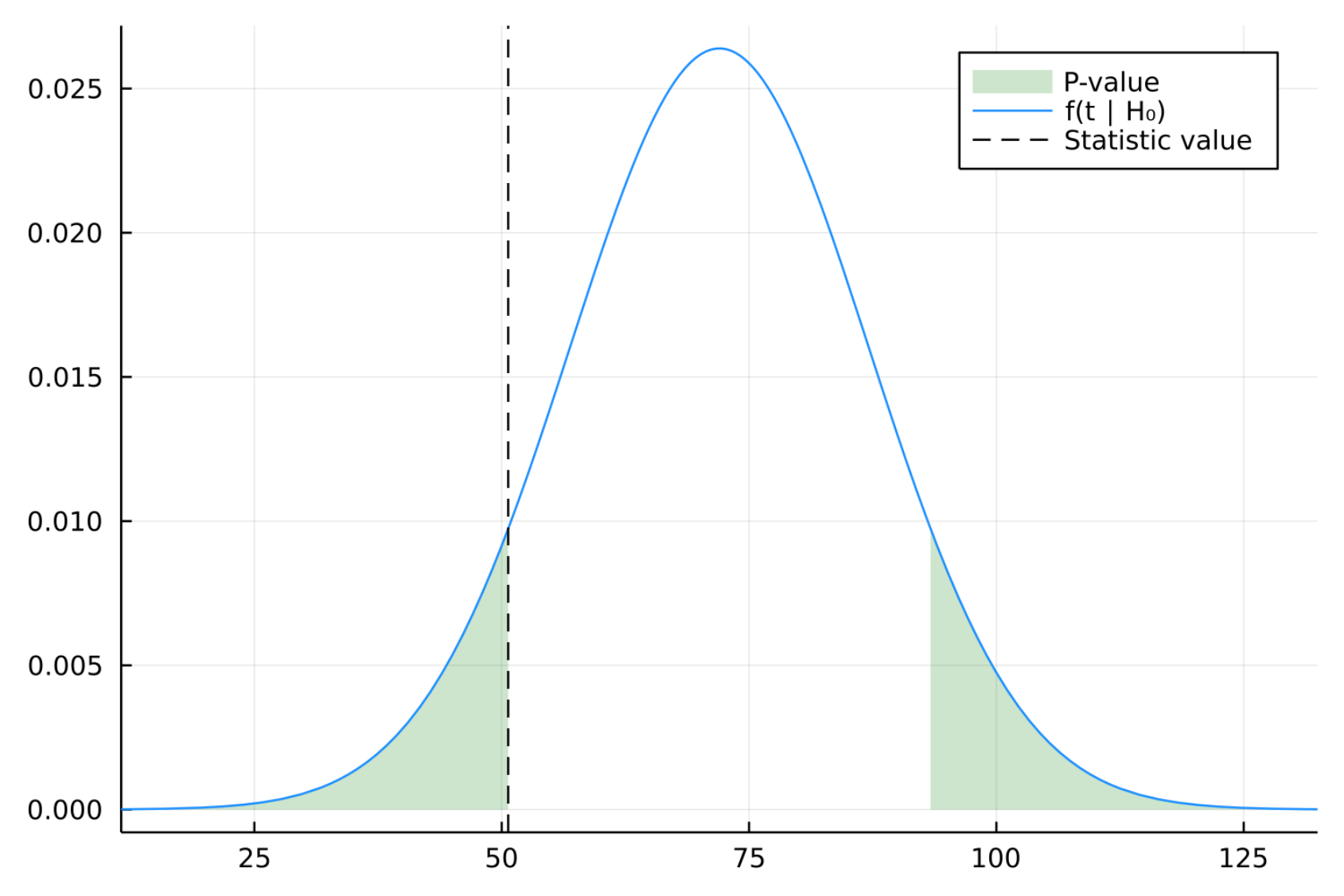
$$\bar{X} = \frac{\sum_{i=1}^{10} X_i}{10} \quad \Rightarrow \quad \bar{X} \mid H_0 \sim \text{Normal}(\mu, \sigma/\sqrt{n}) = \text{Normal}(72, 15.12)$$

Statistic value:

$$\bar{x} = \frac{\sum_{i=1}^{10} x_i}{10} = 50.65.$$

Decision: Assess the congruency between the test statistic under the null hypothesis and the observed statistic value.

Hypothesis testing



P-value = 0.16. Do not reject the null hypothesis.

Hypothesis testing

- **Definition**: It is the process of taking a decision of rejecting or not rejecting a falsifiable statement about a parameter of a population under study.
- **Purpose**: The goal is to make a judgment with respect to the difference between the sample statistics and the parameter.
- **Hypothesis**: Statement about the parameter.
  - **Null hypothesis (H_0)**: Falsifiable statement about a population characteristic (parameter).
  - **Alternative hypothesis (H_1)**: Negative statement of the null hypothesis. Researcher hypothesis.
- **P-value**: The probability of observing a value as extreme as the statistic value under H_0.

$$\text{P-value} = 2\Pr(T < t \mid H_0) \quad \text{or} \quad 2\Pr(T > t \mid H_0).$$

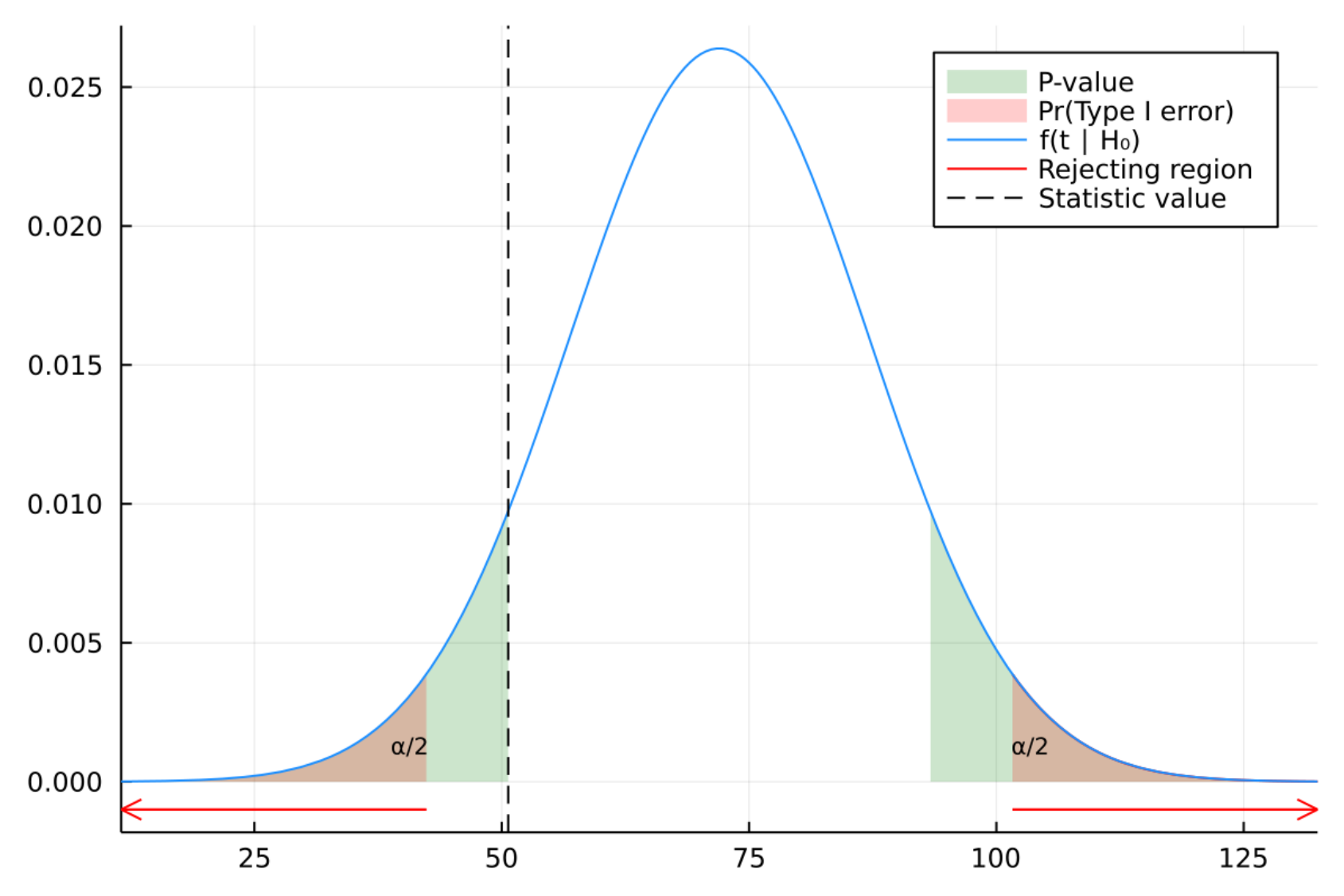
Hypothesis testing

Errors:

- **Type I error** (α): Reject the null hypothesis when this is true. Pr (Type I error) = α.
- **Type II error** (β): Not reject the null hypothesis when this is false. Pr (Type II error) = β.

Level of confidence: $1 -$ Pr (Type I error).

Hypothesis testing



The statistic value 50.65 does not lie in the rejecting region. Do not reject the null hypothesis.

Hypothesis testing

1. Define the null and alternative hypothesis.
2. Define the level of confidence.
3. Define the test statistic and its distribution under the null hypothesis.
4. Compute the statistic value and the p-value (or rejecting region).
5. Take a decision based on the p-value (or rejecting region).

Monte Carlo

Motivation

Distribution for the test statistic

- Obtaining the distribution of the test statistic under the H_0 is essential for hypothesis testing.
- We do not always obtain the distribution of the test statistic.
- Assumptions to obtain the distribution of the test statistic do not always hold.

Monte Carlo for hypothesis testing

We can use Monte Carlo simulation to obtain samples under the null hypothesis (Gentle 2002, 2009).

Monte Carlo for hypothesis testing

We can define the computation of the P-value as an integration problem.

$$\begin{aligned}
\text{P-value} &= 2\Pr\left(T(\boldsymbol{X}) < t \mid H_o\right) \\
&= 2\int_t^{\infty} f_{T|H_o}(z)dz \\
&= 2\int_{T(\boldsymbol{x})<t} f_{\boldsymbol{X}|H_o}(x)d\boldsymbol{x} \\
&= 2\int_{-\infty}^{\infty} I(T(\boldsymbol{X}) < t) f_{\boldsymbol{X}|H_o}(x)d\boldsymbol{x}.
\end{aligned}$$

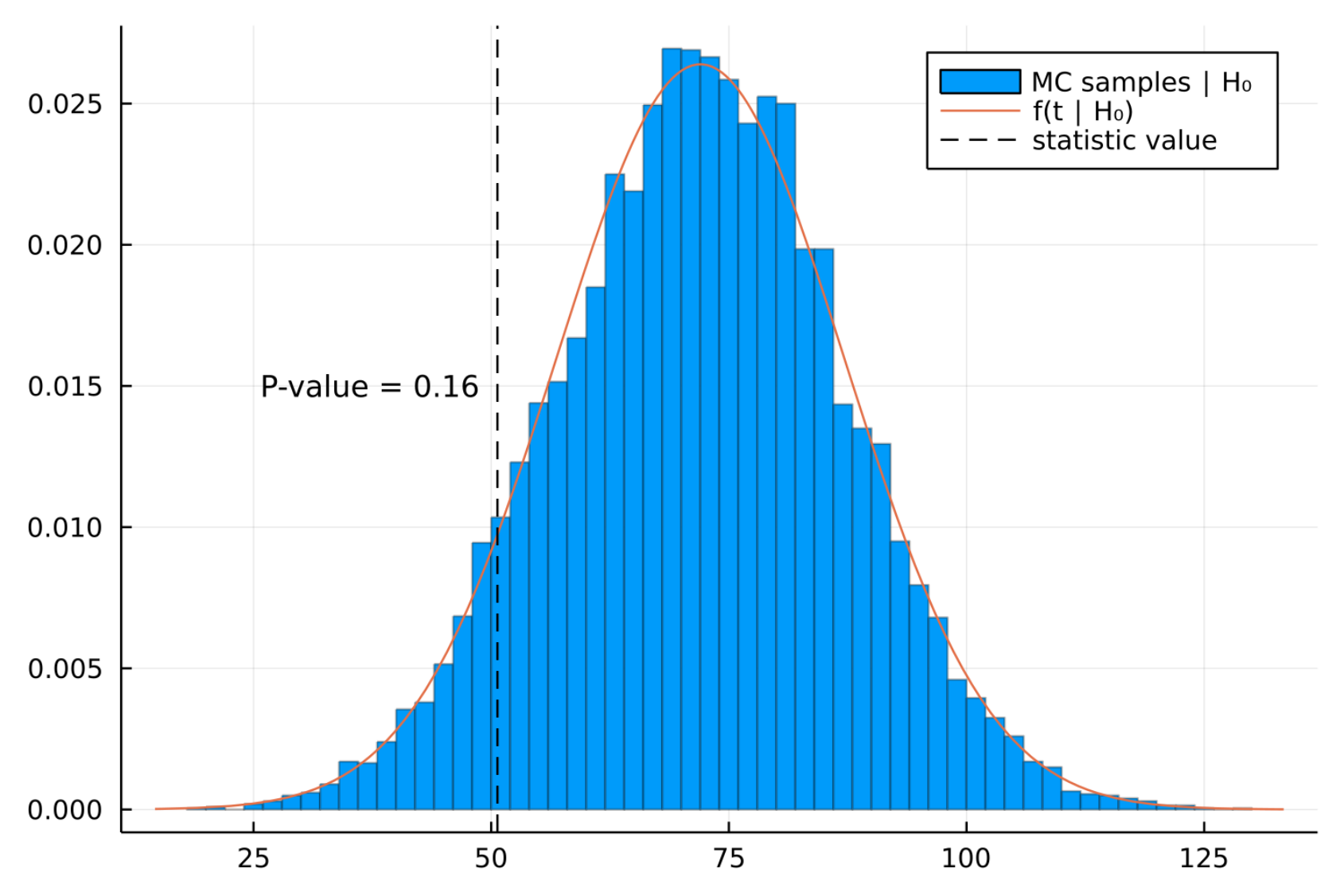
Assuming the random samples $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{t}_m \sim f_{\boldsymbol{X}|H_o}(\cdot)$. Then the Monte Carlo estimator can be defined as

$$\text{P-value}_{MC} = 2\sum \frac{I(T(\boldsymbol{x}_i) < t)}{m}.$$

Monte Carlo for hypothesis testing

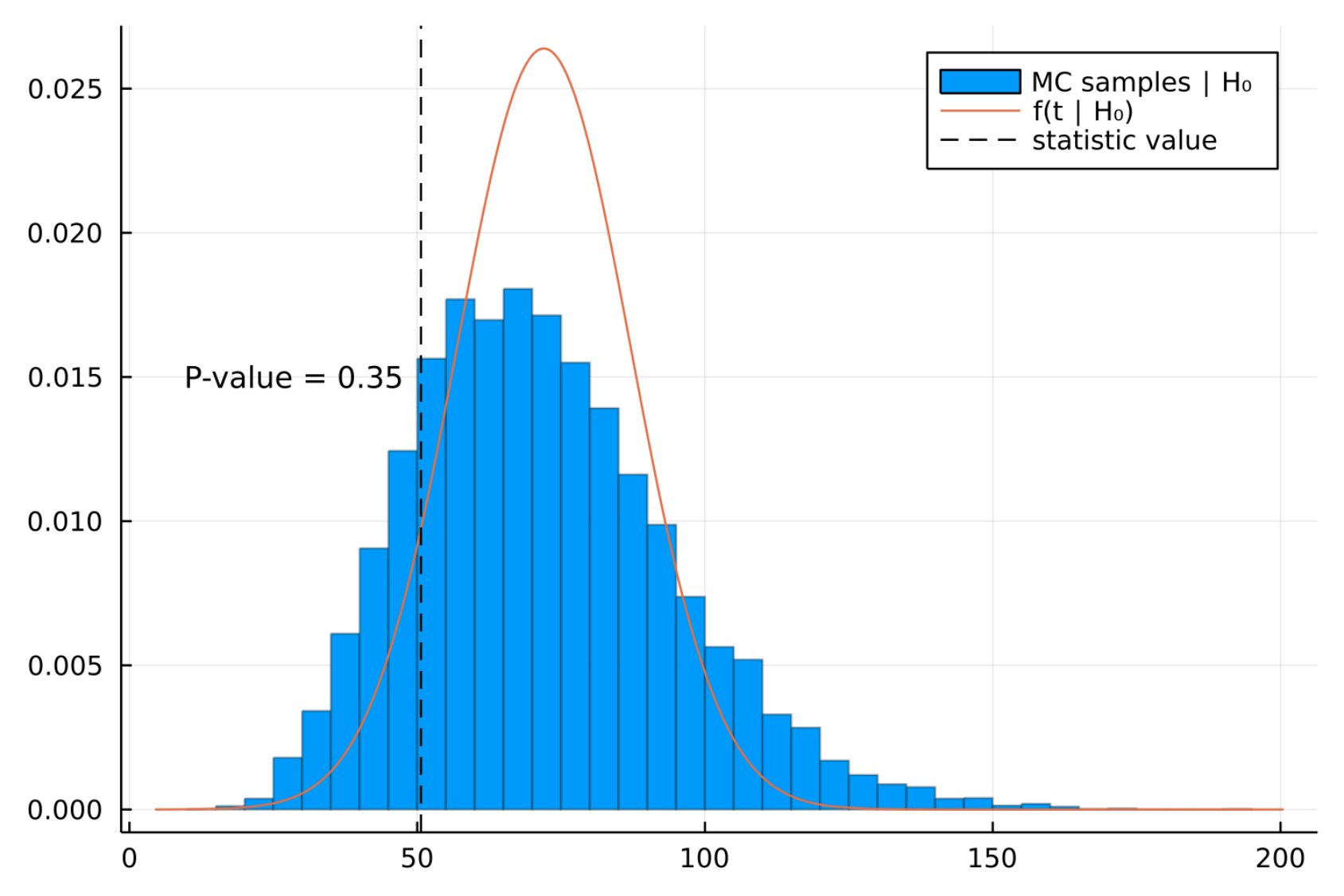
1. Define the null and alternative hypothesis.
2. Define the level of confidence.
3. Define the test statistic and obtain realizations under the null hypothesis.
    3.1 Sample observations $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ under H_0.
    3.2 Compute the statistic value $t_1, \ldots, t_m$.
4. Compute Monte Carlo estimate of the p-value.
5. Take a decision based on the p-value.

Hypothesis testing: assuming normality



P-value = 0.16. Do not reject the null hypothesis.

Hypothesis testing: assuming exponential



P-value = 0.35. Do not reject the null hypothesis.

Tests

Common hypothesis tests

- Mean, mode, median.
- Varianza, standard deviation, correlation.
- Distribution.
- Temporal autocorrelation, spatial autocorrelation.
- Many more.

References

References

Gentle, James E. 2002. *Elements of Computational Statistics*. Springer a.

———. 2009. *Computational Statistics*. Springer Science & Business Media.